



বিদ্যাসাগর বিশ্ববিদ্যালয়
VIDYASAGAR UNIVERSITY
Question Paper

B.Sc. Honours Examination 2022
(Under CBCS Pattern)
Semester - VI
Subject : MATHEMATICS
Paper : C 14 - T

Full Marks : 60

Time : 3 Hours

*Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.*

[RING THEORY AND LINEAR ALGEBRA-II]

1. Answer any *five* questions : 2×5=10
- (a) Show that $1 + \sqrt{-5}$ is irreducible element in $Z(\sqrt{-5})$.
 - (b) Let a, b be two nonzero elements of an Euclidean domain R . If b is not a unit in R , then prove that $d(a) < d(ab)$.
 - (c) Give an example (with justification) of a division ring which is not a field.
 - (d) Determine all the associates of $[8]$ in the ring \mathbb{Z}_{10} .
 - (e) Give an example of a linear operator T on a finite dimensional vector space V over a field F such that T is not diagonalizable.

- (f) Find the *dual basis* of the basis $\{(1, 1, 2), (1, 0, 1), (2, 1, 0)\}$ of the vector space \mathbb{R}^3 .
- (g) If a real symmetric matrix is positive definite then show that all its eigen values are positive.
- (h) If $T \in A(V)$ and S is regular in $A(V)$, prove that T and STS^{-1} have same minimal polynomial, where $A(V)$ is the annihilator of V .

2. Answer any **four** questions :

5×4=20

- (a) (i) Prove that 1 and -1 are the only units of the ring $\mathbb{Z}\sqrt{-5}$.
- (ii) Show that the integral domain $\mathbb{Z}\sqrt{-5}$ is a factorization domain. 3+2=5
- (b) Find *gcd* of $11 + 7i$ and $18 - i$ in $Z + iZ$.
- (c) Let $T : V \rightarrow V$ be a linear mapping, where V is a Euclidean space. Show that T is orthogonal if and only if T maps an orthogonal basis to an orthonormal basis.
- (d) Let V be a finite dimensional vector space over the field F and T be a diagonalizable linear operator on V . Let $\{c_1, c_2, \dots, c_k\}$ be the set of all distinct eigen values of T . Then prove that the characteristic polynomial of T is of the form $(x - c_1)^{d_1} (x - c_2)^{d_2} \dots (x - c_k)^{d_k}$ for some positive integers d_1, d_2, \dots, d_k .
- (e) (i) Let T be a linear operator on a finite dimensional vector space V over F . Define minimal polynomial of T .
- (ii) If V is finite dimensional over F , then prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial of T is not 0, where $A(V)$ is the annihilator of V . 1+4
- (f) Find the eigen values and bases for the eigen space of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$.

Is A diagonalizable?

3. Answer any **three** questions :

10×3=30

- (a) (i) Let R be a PID. Prove that p is irreducible in R if and only if the ideal generated by p is a non-zero maximal ideal. Hence show that $\mathbb{Q}[x]/\langle x^2 - 2 \rangle$ is a field.
- (ii) Prove that for any linear operator T on a finite-dimensional inner product space V , there exists a unique linear operator T^* on V such that $\langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle$ for all $\alpha, \beta \in V$. (4+2)+4=10
- (b) (i) Let N be a 2×2 complex matrix such that $N^2 = 0$. Then prove that either $N = 0$ or N is similar to the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ over \mathbb{C} .
- (ii) Use Gram-Schmidt process to obtain an orthonormal basis from the following basis $\mathcal{B} = \{ (1, 2, -2), (2, 0, 1), (1, 1, 0) \}$ of \mathbb{R}^3 with the standard inner product. 4+6=10
- (c) (i) Show that an element x in a Euclidean domain is a unit if and only if $d(x) = d(1)$. Hence find all units in the ring $\mathbb{Z} + i\mathbb{Z}$ of Gaussian integers.
- (ii) Define unique factorization domain (UFD). Show that $R = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ is not UFD. 6+4
- (d) (i) Consider the polynomial $f(x) = 5x^4 + 4x^3 - 6x^2 - 14x + 2$ in $\mathbb{Z}[x]$. Using Eisenstein's criterion show that $f(x)$ is irreducible in \mathbb{Z} .
- (ii) Let $A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}$. Find its minimal polynomial over \mathbb{R} and hence check whether A is similar to a diagonal matrix or not.
- (iii) Consider the inner product space \mathbb{C}^2 over \mathbb{C} with the standard inner product. Let T be a linear operator on \mathbb{C}^2 such that the matrix representation of T with respect to the standard ordered basis is $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Show that T is a normal operator. 3+(3+1)+3=10

(e) (i) Let a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by

$T(x, y, z) = (2x + y - 2z, 2x + 3y - 4z, x + y - z)$. Find all eigen values of T and find a basis of each eigen space.

(ii) The matrix of a linear mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ relative to the standard basis

is $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}$. Find f and its matrix with respect to the basis

$\{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$. 5+5
