বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : GE 4 - T

Full Marks : 40

Time : 2 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

[NUMERICAL METHODS]

1. Answer any ***four*** questions : 5×4=20

(a) (i) Deduce Newton-Cotes quadrature formula.

(ii) Evaluate : $\left(\frac{\Delta^2}{E}\right)x^3$ 3+2

(b) Given $(n + 1)$ distinct points $x_0, x_1, x_2, ..., x_n$ and $(n + 1)$ ordinates $y_0, y_1, ..., y_n$, there is a polynomial $p(x)$ of degree $\leq n$ that interpolates to y_i at $x_i, i = 0,1,....,n$. Prove that this polynomial is unique.

(c) Describe the Regula-Falsi method for finding the root of the equation $f(x) = 0$. What are the advantages and disadvantages of this method.

(d) Let *f(x)* be a function. Describe least square method to approxiate a polynomial.

(e) Describe Gauss-elimination method for numerical solution of a system of linear equations.

(f) Evaluate y (1.0) from the differential equation $\frac{dy}{dx} = y + x^2$ with $y(0)=1$ taking $h=0.2$, by Euler's method correct upto two decimal places.

2. Answer any ***two*** questions : 10×2=20

(a) (i) Derive the Simpson $\frac{1}{3}$ integration formula in the form

$$\int_a^b f(x)\,dx = \frac{b-a}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{b-a}{2^5 \times 90} f^{(iv)}(g)$$

where $a<g<b$. What is the error if *f(x)* is a polynomial of degree 3. 6

(ii) Find the value of $\int_0^1 \frac{1}{1+x}\,dx$ using Simpson's $\frac{1}{3}$ rule and mid-point formula using $h = 0.5$. 4

(b) Derive the convergence criteria for Newton-Raphson method. Also determine the order of convergence of this method. 5+5

(c) Describe power method to find the largest magnitude eigen value of a square matrix.

(d) (i) Solve the following system of equations by Gauss-seidal iteration method correct upto three significant figures :

$3x + y + z = 3;\ 2x + y + 5z = 5;\ x + 4y + z = 2$ 7

(ii) Compute the percentage error in the time period $T = 2\pi\sqrt{\frac{l}{g}}$ for $l = 1m$ if the error in the measurement of l is 0·01. 3

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OR

[PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS]

Full Marks : 60

Time : 3 Hours

1. Answer any ***five*** questions : 2×5=10

(i) Find the order and degree of the following PDE :
$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$

(ii) Form a PDE by the elimination of the arbitrary constants a, b from $z = ax + by$.

(iii) Determine whether the equation $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} = 0$ is hyperbolic, parabolic or elliptic.

(iv) Write and classify Laplace's equation.

(v) Give an example of a homogeneous linear second order PDE.

(vi) State Kepler's second law.

(vii) Write the Lagrange's auxiliary equations for the PDE $zxp + zyq = xy$.

(viii) A particle describes the curve $p^2 = ar$ under a force F to the pole. Find the law of force.

2. Answer any ***four*** questions : 5×4=20

(i) Form a PDE by eliminating the function *f* from $z = f\left(x^2 - y^2\right)$.

(ii) Using Lagrange's method solve the PDE $\left(y+z\right)p + \left(z+x\right)q = x + y$.

(iii) Show that the characteristics equation of the PDE $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$ represents a family of straight lines passing through the origin.

(iv) Find the complete integral of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=au+\frac{xy}{z}$.

(v) A particle describes a curve whose equation is $r=a\sec^2\frac{\theta}{2}$ under a force to the pole. Find the law of force.

(vi) A particle describes the path $r=a\tan\theta$ under a force to the origin. Find its acceleration in terms of r.

3. Answer any ***three*** questions : 10×3=30

(i) Transform the partial differential equation $\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}=0$ to cannonical form and hence solve it.

(ii) Apply the method of separation of variables to obtain a formal solution $u(x,y)$ of the problem which consists of the wave equation $\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$ with the conditions :

$$u(0,y)=u(\pi,y)=0, y\geq 0$$

$$u(x,0)=\sin 2x, 0\leq x\leq \pi$$

$$\frac{\partial u(x,0)}{\partial y}=0, 0\leq x\leq \pi$$

(iii) Find the solution of the initial boundary value problem :

$$u_{tt}=u_{xx}, 0<x<2, t>0$$

$$u(x,0)=\sin\left(\frac{\pi x}{2}\right), 0\leq x\leq 2$$

$$u_t(x,0)=0, 0\leq x\leq 2$$

$$u(0,t)=0,\ u(2,t)=0, t\geq 0$$

(iv) Find the solution of the cauchy problem for the first order PDE $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ on $D=\{(x,y,z): x^2+y^2\neq 0, z>0\}$ with the initial condition $x^2+y^2=1,\ z=1$.

(v) Show that the path described under the inverse square law of distance will be an ellipse, a parabola or a hyperbola according as $v^2 <=\text{ or } > \frac{2\mu}{r}$.

OR

[RING THEORY AND LINEAR ALGEBRA-I]

Full Marks : 60

Time : 3 Hours

1. Answer any ***five*** questions : 2×5=10

   (a) Let D be an integral domain and $a, b \in D$. If $a^5 = b^5$ and $a^8 = b^8$, prove that $a = b$.

   (b) Examine whether the mapping $T : R^2 \to R^3$ defined by $T(x, y) = (x + 2y, 2x + y, x + y), (x, y) \in R^2$ is a linear mapping.

   (c) Examine if the set = $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$, is a subspace of $\mathbb{R}^3$.

   (d) Prove that in a ring R if a is an idempotent element then $1 - a$ is also idempotent.

   (e) Prove that $\mathbb{Z}$ and $2\mathbb{Z}$ are not isomorphic.

   (f) In a ring R, prove that (i) $(-a)(-b) = ab$, (ii) $a(b - c) = ab - ac$ for all $a, b, c \in R$.

   (g) Show that the set $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$ of diagonal matrices is a subring of the ring of all 2 × 2 matrices over $\mathbb{Z}$.

   (h) Is $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + 2x_3 = 0\}$ a sub-space of $\mathbb{R}^3$? Justify.

2. Answer any ***four*** questions : 5×4=20

   (a) Prove that the set $Z\sqrt{-5} = \{a + b\sqrt{-5} : a, b \in Z\}$ is an integral domain with usual addition '+' and multiplication '**.**' of two complex number.

   (b) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$, $(x, y, z) \in \mathbb{R}^3$. Find the matrix of T relative to the order bases ((0,1,1), (1,0,1), (1,1,0)) of R^3 and ((1,0), (0,1)) of $\mathbb{R}^2$.

(c) Let R be a commutative ring and suppose $nx = 0 \, \forall x \in R$ where n is a prime number. Then show that the mapping $f : R \to R$ defined by $f(x) = x^n,\ x \in R$ is a homomorphism.

(d) Suppose $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a basis of a vector space V over a field F and a non zero vector β of V is expressed as $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 : c_i \in F$, i = 1,2,3,4 then if $c_4 \neq 0$, then prove that $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ is a new basis.

(e) Show that a ring R is commutative iff $(a+b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$. Show that Z_p modulo p is a field if and only if p is a prime. 2+3

(f) Show that the vectors $v_1 = (0, 2, -4), v_2 = (1, -1, 1), v_3 = (1, 2, 1)$ are linearly independent in $\mathbb{R}^3(\mathbb{R})$. If $\alpha, \beta, \gamma \in V(F)$ such that $\alpha + \beta + 2\gamma = 0$, then show that $\{\alpha, \beta\}$ spans the same subspace as $\{\beta, \gamma\}$ i.e., show that $L(\{\alpha, \beta\}) = L(\{\beta, \gamma\})$. 2+3

3. Answer any ***three*** questions : 10×3=30

(a) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ that maps the basis vectors (0,1,1), (1,0,1), (1,1,0) of $\mathbb{R}^3$ to the vectors (2, 1, 1), (1, 2, 1), (1, 1, 2) respectively. Find *Ker T* and *Im T*. Verify that dim *Ker T* + dim *Im T* = 3. 4+2+2+2

(b) (i) Prove that a commutative ring R with unity is an integral domain if and only if for every non-zero element a in R, $a.u = a.v \Rightarrow u = v$, where $u, v \in R$.

(ii) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (2x + z, x + y + z, -3x - z), (x, y, z) \in \mathbb{R}^3$. Show that T is an isomorphism. 6+4

(c) (i) Prove that in the ring of integers $(\mathbb{Z}, + \cdot)$ every ideal is a principal ideal.

(ii) Let X be a non empty set. Show that the $P(X)$, the power set of X forms a commutative ring with unity under $\oplus$ and $\odot$ defined by $A \oplus B = (A \cup B) - (A \cap B)$ and $A \odot B = A \cap B$ where $A, B \in P(X)$. 5+5

(d) Show that in an integral domain R (with unity) the only idempotents are the zero and unity. If A is an ideal of a ring R with unity such that $1 \in A$ then show that $A = R$. Determine all the ideals of the ring of integers $(Z, +, \cdot)$. Show by an example that it is possible to have a ring R with unity where $\{0\}$ and R are the only ideals of R, but R is not a division ring. 3+2+3+2

(e) Prove that $L(S)$ is the smallest subspace of V, containing S. If $f : R \to R'$ be an onto homomorphism, then R' is isomorphic to a quotient ring of R. In fact $R' \cong \frac{R}{Ker\, f}$.

Show that $\frac{\mathbb{Z}}{<2>} = \frac{5\mathbb{Z}}{10\mathbb{Z}}$. 3+4+3

OR

[MULTIVARIATE CALCULUS]

Full Marks : 60

Time : 3 Hours

1. Answer any ***five*** questions : 2×5=10

(a) If $f(x,y)=\dfrac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)}\dfrac{xy}{x^2+y^2}$ exist?

(b) Find the extreme values of $f(x,y)=y^2-x^2$.

(c) Evaluate $\iint_R e^{-(x+y)}dxdy$ where R is the region in the first quadrant in which $x+y\le 1$.

(d) Find the maximum rate of change of the function $f(x,y)=\sqrt{x^2+y^4}$ at the point (–2, 3) and the direction in which this maximum rate of change occurs.

(e) Convert the integral to cylindrical coordinates : $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} x\,dz\,dy\,dx$.

(f) Find the tangent plane to the elliptic paraboloid $z=2x^2+y^2$ at the point (1,1,3).

(g) Where is the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ continuous?

(h) Evaluate $\int_C xy\,dx+x^2dy$, where C is given by $y=x^3, -1\le x\le 2$.

2. Answer any ***four*** questions : 5×4=20

(a) Define chain rule for functions involving two independent variables. If $g(s,t)=f\left(s^2-t^2,t^2-s^2\right)$ and f is differentiable, show that g satisfies the equation $t\dfrac{\partial g}{\partial s}+s\dfrac{\partial g}{\partial t}=0$. 2+3

(b) Define the gradient of the function $f(x, y)$. Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point (2, –1) in the direction of the vector $\overline{v} = 2\hat{i} + 5\hat{j}$. 1+4

(c) Show that the line integral $\int_C (y + yz)dx + (x + 3z^3 + xz)dy + (9yz^2 + xy - 1)dz$.

is independent of the path C between (1,1,1) to (2,1,4). 5

(d) State sufficient condition for differentiability. Show that $f(x, y) = x^2y + xy^3$ is differentiable for all (x, y). 2+3

(e) Use a polar double integral to show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$. 5

(f) If $\overline{F}(x, y, z)$ be a continuously differentiable vector function, then prove that

$\overline{\nabla} \times (\overline{\nabla} \times \overline{F}) = \overline{\nabla}(\overline{\nabla} \cdot \overline{F}) - \overline{\nabla}^2 \overline{F}$ 5

3. Answer any ***three*** questions : 10×3=30

(a) (i) State and prove Stoke's theorem for curls.

(ii) Define total differential of a function *f(x, y, z)*. Determine the total differential of the function $f(x, y) = x^2 \ell n (3y^2 - 2x)$. 6+(2+2)

(b) (i) State Young's theorem.

(ii) Consider the function f defined by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & where\ x^2 + y^2 \neq 0 \\ 0, & where\ x^2 + y^2 = 0 \end{cases}$$

Show that $f_{xy} \neq f_{yx}$ at (0, 0).

(iii) Let R be the annular region lying between the two circles $x^2+y^2=1$ and $x^2+y^2=5$. Evaluate the integral $\iint_R (x^2+y)\,dA$. (1+5)+4

(c) (i) Changing the order of integration, show that

$$\int_0^1 dx \int_x^{\frac{1}{x}} \frac{y\,dy}{(1+xy)^2(1+y^2)} = \frac{\pi-1}{4}.$$

(ii) Evaluate $\iiint xyz\,dx\,dy\,dz$ over the region $R: 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

6+4

(d) (i) State Green's Theorem in the plane.

(ii) Show that $\bar{F}(x,y,z) = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$ is irrotational. Find a scalar function φ such that $\bar{F} = \bar{\nabla}\varphi$.

(iii) Find the work done by the force $\bar{F}(x,y) = (-16y+\sin x^2)\hat{i} + (4e^y+3x^2)\hat{j}$ acting along the simple closed curve $C: x^2+y^2=1, y=x, y=-x$. 2+4+4

(e) (i) State the Gauss's Divergence theorem. Verify Gauss's divergence theorem for $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2+y^2=4, z=0$ and $z=3$.

(ii) Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (x\,dy - y\,dx)$. (1+6)+3

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