2022

5th Semester Examination

MATHEMATICS (Honours)

Paper : DSE 2-T

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Probability and Statistics]

Group - A

1. Answer any *ten* questions : 2×10=20

(a) A freshman class at a college has 200 students of which 150 are women and 50 are majoring in maths, and 25 maths major are women. If a student is selected at random from the freshman class, what is the probability that the student will be either a mathematics major or a women?

(b) A speaks the truth in 75% cases and B in 80% cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

(c) Write the pdf of Gamma distribution and its mean and variance.

(d) Find $E(X)$ for the following density function :

$$f(x)=\begin{cases}\frac{4x}{5}, & 0<x\leq 1\\ \frac{2}{5}(3-x), & 1<x\leq 2\\ 0, & elsewhere.\end{cases}$$

(e) Accidents take place in a factory at a rate of 6 per year. What is the probability that there is no accident in a given month?

(f) X, Y, Z are three random variables, with $\sigma_x=2$, $\sigma_y=1$ and $\sigma_z=3$; $\rho_{xy}=0.3$, $\rho_{yz}=0.5$ and $\rho_{zx}=0.5$. Find the variance of $U=X+Y-Z$.

(g) If the lines of regression of y on x and x on y are $3x+2y=26$ and $6x+y=31$, respectively. Find the correlation coefficient between x and y.

(h) Let U and V be two random variables with $E(U)=0=E(V)$, $\text{var}(U)=\text{var}(V)=1$. Then prove that $-1\leq E(UV)\leq 1$.

(i) State weak and strong law of large numbers.

(j) Let $X=(X_1, X_2, \cdots, X_{54})$ be a random sample from a discrete distribution with pmf $p(x)=\frac{1}{3}$, $x=2, 4, 6$. Find the probability distribution of sample mean $\bar{X}$ using central limit theorem.

(k) Let $X_1, X_2, \cdots, X_n$ be independent and identically $N(\mu, \sigma^2)$ distributed. Find method of moment estimator of μ, σ^2.

(l) The bivariate random variable (X, Y) jointly follow the probability density function

$$f(x,y)=\begin{cases}kx^2(8-y), & x<y<2x, 0\leq x\leq 2\\ 0, & elsewhere.\end{cases}$$

Find the k.

(m) Let $X_1, X_2, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Find the sampling distribution of $W=\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$.

(n) Let X be a random variable follows $N(800, 144)$ distribution. Find $P(X<772)$. Given that $P(Z<2.33)=0.0099$, where Z follows standard normal distribution.

(o) Define Markov chain with an example.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Let $X \sim Bin(n, p)$ and $Y = \frac{X-np}{\sqrt{npq}}$. Prove that the distribution of Y converges to $N(0, 1)$ as $n \to \infty$ (not using Central limit theorem).

(b) State and prove Chapman-Kolmogorov equation.

(c) Let $X_1, X_2, \cdots, X_n$ be independent and identically $N(\mu, \sigma^2)$ distributed. Find method of moment estimator of μ, σ^2 by calculating raw moments.

(d) Find the value of k so that the following table may represent a joint distribution

	$Y=1$	$Y=2$
$X=1$	0.4	0.1
$X=2$	k	0.3

Find conditional distribution of X given $Y=y$ and also find conditional expectation of X given $Y=y$.

(e) A die is thrown 3600 times, show that the probability that the number of sixes lies between 550 and 650 is at least 4/5 (use Chebyshev's inequality).

(f) Bearings made from a certain process have a mean diameter 0.0566 cm and a standard deviation 0.004 cm. Assuming that the data may be looked upon as a random sample from a normal population, construct a 95% confidence interval for the actual average diameter of bearings made by the process. Given that $P(t > 2.262) = 0.025$ with 9 degrees of freedom and $P(t > 2.228) = 0.025$ with 10 degrees of freedom.

Group - C

3. Answer any ***two*** questions : 10×2=20

(a) (i) What is called likelihood function?

(ii) Let $X_1, X_2, \cdots, X_n \sim U(a, b)$. Find maximum likelihood estimators of a and b.

(iii) A random sample of size 25 is taken from a Poisson distribution with the parameter λ. If the sum of all observations is 150, what is the method of moment estimate of λ? 2+3+5

(b) Following are the mileages recorded (km per litre of petrol) in 16 runs of a new model of car : 22.16, 22.37, 22.50, 22.04, 22.25, 23.01, 22.81, 22.63, 23.18, 22.55, 22.75, 22.95, 22.50, 22.38, 23, 22.17.

Assuming the mileage follows a normal distribution with mean μ and variance σ^2, test the hypotheses

(i) $H:\mu = 22.5$ vs. $H_1:\mu \neq 22.5$ and

(ii) $H:\sigma^2 \leq 0.3$ vs. $H_1:\sigma^2 > 0.3$.

Take level of significance 0.05.

Given that $t_{0.025,\,15} = 2.131$, $t_{0.05,\,15} = 1.753$, $\chi^2_{0.05,15} = 24.996$, $\chi^2_{0.025,15} = 27.488$, choose the appropriate. 5+5

(c) The joint density function of (X, Y) is given by $f(x,y) = \begin{cases} 10xy^2, 0 < x < y < 1 \\ 0, \quad elsewhere \end{cases}$. Find the marginal and conditional probability density functions of X and Y. Also find $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, $\text{cov}(X,Y)$ and $\rho(X,Y)$. 10

(d) (i) Let $F(x)$ be the distribution function of a continuous random variable X. Show that the expectation of X can be expressed as $E(X) = \int_{x=0}^{\infty}\{1 - F(x) - F(-x)\}\,dx$.

(ii) For any random variable X (discrete or continuous) and for any real number c, prove

that $E(|X-c|) \geq E(|X-\mu|)$ provided the expectations exist and μ is the median of X.

(iii) If X is $\gamma(l)$ variate, then compute $E(\sqrt{X})$.

4+4+2

OR

[Boolean Algebra and Automata Theory]

1. Answer any ***ten*** of the following : 2×10=20

   (a) Show that the relation $\leq$ is a total order on the set of real numbers **R**.

   (b) Define Strict and Partial Orders in a set.

   (c) Identify extreme elements in the Poset : "*The divisors of 60, ordered by divisibility.*"

   (d) Is D_{12} a Boolean lattice? Explain.

   (e) Let $< A, \leq >$ be a totally ordered set. Prove that if A has more than two elements, then it is not a complemented lattice, even if it has a minimum and a maximum.

   (f) Prove that every finite Boolean lattice with more than one element has atomic elements.

   (g) Prove the following proposition, using the axioms of Boolean Algebra : $x(x+y)=x$.

   (h) Show how *AND* can be simulated using only *NAND* gates.

   (i) Calculate the number of distinct Boolean functions from B^n to B.

   (j) Define empty string and the length of a string.

   (k) How a *DFA* processes string?

   (l) What is transition diagram for *DFA*?

   (m) Differentiate between *DFA* and *NFA*.

   (n) Define pumping lemma for regular languages.

   (o) Convert the grammar $A \rightarrow aS | bS | a$ to a PDA that accepts the same language by empty stack.

2. Answer any ***four*** of the following : 5×4=20

   (a) Prove that a Language L is accepted by some *DFA* if and only if L is accepted by some *NFA*.

   (b) Design a PDA to accept each of the following languages :

      (i) $\{a^i b^j c^k \mid i \neq j \text{ or } j \neq k\}$

      (ii) The set of all strings of a's and b's that are not of the form ww, i.e., not equal to any string repeated.

   (c) If $L = N(P)$ for some DPDA, P, then show that L has a unambiguous context free grammar.

   (d) Show that L_{ne} is recursively enumerable.

   (e) Use Karnaugh maps to find the minimal form for the expression : $xyz + xyz' + xy'z + x'yz + x'y'z$.

   (f) The Boolean function $Y = AB + CD$ is to be realized using only 2 input NAND gates. What is the minimum number of gates required?

3. Answer any ***two*** of the following : 10×2=20

(a) (i) Convert to a *DFA* the following *NFA* and informally describe the language it accepts : 6

	0	1
$\rightarrow p$	$\{p,q\}$	$\{p\}$
q	$\{r\}$	$\{r\}$
r	$\{s\}$	θ
$*s$	$\{s\}$	$\{s\}$

(ii) If L and M are regular languages then show that $L \cap M$ and L^R are also regular languages. 4

(b) (i) Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDA then show that there exist a context free grammar G such that $L(G) = N(P)$. 6

(ii) Design the turning machine for the following language : $\{a^n b^n c^n \mid n \geq 1\}$. 4

(c) (i) Show that the divisibility relation is not a partial order on the set of integers Z. Which property is lacking? 3

(ii) Show that every non-empty subset of a poset is also a poset. 3

(iii) Give an example to show that maximal and minimal elements of S need not be unique. 2

(iv) If a poset is infinite, can it be embedded in a totally ordered set? Prove it or disprove it. 2

(d) (i) State and prove the De-Morgans law for Boolean lattice. 6

(ii) Show that if a Boolean lattice has more than two elements then it is not totally ordered. 2

(iii) Define Boolean algebra. 2

OR

[Portfolio Optimization]

1. Answer any *ten* questions : 2×10=20

   (a) What is the Portfolio Management Process?

   (b) Explain the structure of SEBI.

   (c) What are the Types of Investors?

   (d) How would you calculate the cost of Equity?

   (e) What is the monetary policy?

   (f) What are the tax benefits in mutual fund?

   (g) Which is better Equity or Real Estate?

   (h) What is NAV?

   (i) You save Rs. 100 and invest it at a nominal interest rate of 8%. Given the expected inflation is 5% per year, what is the real rate of return?

   (j) What is portfolio risk and return?

   (k) What is Annuity?

   (l) Differentiate between Security Market Line (SML) and Capital Market Line (CML).

   (m) Define diversification.

   (n) Explain Rebalancing.

   (o) What is a primary market?

2. Answer any *four* questions : 5×4=20

   (a) Define :

   (i) Beta of portfolio

   (ii) Security market line

   (b) You have a portfolio with a beta of 0.84. What will be the new portfolio beta if you keep 85% of your money in the old portfolio and 14% in a stock with a beta of 1.93?

   (c) What are some of the benefits of diversification?

   (d) Use the information in the following to answer the questions below :

State of Economy	Probability of state	Return on A in state	Return on B in state
Boom	35%	0.040	0.210
Normal	50%	0.030	0.080
Recession	15%	0.046	–0.010

   What is the expected return of each asset?

   (e) What are the functions of SEBI?

   (f) How do Mutual Funds work?

3. Answer any *two* questions : 10×2=20

   (a) Prove that the expected return μ_i on any asset i satisfies $\mu_i = r_f + \beta_i(\mu_M - r_f)$, where $\beta_i = \frac{\sigma_{iM}}{\sigma_{M^2}}$

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and σ_{iM} is the covariance of the return on asset i and the market portfolio r_M; $\sigma_{M^2} = \text{var}(r_M)$.

(b) Consider 3 assets with rates of return r_1, r_2 and r_3, respectively. The covariance matrix and expected rates of return are $\Sigma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and

$$m = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.8 \end{pmatrix}.$$

(i) Find the minimum variance portfolio.

(ii) Find a second efficient portfolio.

(iii) If the risk free rate is $r_f = 0.2$, find an efficient portfolio of risky assets.

(c) For the Markowitz mean-variance portfolio solve the quadratic programming problem

minimize $\frac{1}{2} w^T \Sigma w - \lambda m^T w$

subject to $e^T w = 1$,

where $w = (w_1, w_2, \ldots \ldots w_n)^T$,
$m = (m_1, m_2, \ldots \ldots m_n)^T$, $\mu_i = E(r_i)$,
$z = (r_1, r_2, \ldots \ldots r_n)^T$, $\text{cov}(z) = \Sigma$

(d) Assume that the expected rate of return on the market portfolio is 24% ($r_M = 0.24$) and the rate of return on T-Bills (risk free rate) is 7% ($r_f = 0.07$). The standard deviation of the market is 33% $(\sigma_M = 0.33)$. Assume that the market portfolio is efficient.

(i) What is the equation for the capital market line?

(ii) If an expected return of 38% is desired, what is the standard deviation of this position?
